

## ANALYTIC TORSION AND SYMPLECTIC VOLUME

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ABSTRACT. This article studies the abelian analytic torsion on a closed, oriented, quasi-regular Sasakian three-manifold and identifies this quantity as a specific multiple of the natural unit symplectic volume form on the moduli space of flat abelian connections. This identification effectively computes the analytic torsion explicitly in terms of Seifert data for a given quasi-regular Sasakian structure on a three-manifold.

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## 1. INTRODUCTION

This article studies the abelian analytic torsion  $T_X$  on a quasi-regular Sasakian three-manifold  $(X, \phi, \xi, \kappa, g)$ . There is also a combinatorially defined analogue of  $T_X$  called the Reidemeister torsion (R-torsion), denoted as  $\tau_X$ . It is a well known fact that  $\tau_X = T_X$ , and we will therefore sometimes refer to the Reidemeister-Ray-Singer torsion generally. Recall that this equivalence was independently shown by Müller, [Mül78], and Cheeger, [Che79], for unimodular representations and more recently an elegant new proof has been given by Braverman [Bra02] using the Witten laplacian [Wit82]. We work with the analytic Ray-Singer torsion directly in this article. We note that a computation using the purely combinatorial definition is lacking and would be very useful complementary approach. For the combinatorial computation, we suggest following a similar approach to [Wit91] where the R-torsion is studied at an irreducible flat  $G$ -connection on a surface, for  $G$  a compact, simply connected, semi-simple, and connected Lie group.

The analytic torsion is a classical three-manifold topological invariant [Rei35], [RS73] and our goal is to prove that its square-root can be identified as a natural symplectic volume form on the moduli space of flat abelian connections in the case that  $X$  admits a Sasakian structure. This identification is motivated by physics arguments involving the abelian Chern-Simons partition function. The basic physics idea is that one is able to start with the same heuristic definition of an abelian Chern-Simons partition function and then use completely different techniques to define it rigorously. Using Faddeev-Popov gauge fixing [FP67] on the one hand and non-abelian localization on the other hand following the ideas of Chris Beasley

and Edward Witten [BW05] one may obtain two a priori different definitions.

Before presenting the relevant definitions we establish some notation and terminology. Throughout,  $X$  will denote a closed, orientable three-manifold and  $(X, \phi, \xi, \kappa, g)$  will denote  $X$  equipped with a (quasi-regular) Sasakian structure. See appendix A for further background on Sasakian and contact geometry. For convenience we recall that a *Sasakian manifold* is a normal contact metric manifold,  $(X, \phi, \xi, \kappa, g)$ , where,

- $\kappa \in \Omega^1(X)$  is a contact form, i.e.  $\kappa \wedge d\kappa \neq 0$ ,  $\xi$  is the Reeb vector field,
- $\phi \in \text{End}(TX)$ ,  $\phi(Y) =: JY$  for  $Y \in \Gamma(H)$ ,  $\phi(\xi) = 0$  where  $J \in \text{End}(H)$  is an almost complex structure on the contact distribution  $H := \ker \kappa \subset TX$ , and,
- $g = \kappa \otimes \kappa + d\kappa \circ (\mathbb{I} \otimes \phi)$ .

Let  $\mathbb{T}$  denote a compact, connected abelian Lie group of dimension  $N$ ,  $\mathfrak{t}$  denote its Lie algebra and  $\Lambda \subset \mathfrak{t}$  the integral lattice. Let  $\text{Tors } H^2(X, \Lambda)$  denote the torsion subgroup of  $H^2(X, \Lambda)$ .  $\mathcal{A}_P$  is the affine space of connections on  $P$  modeled on the vector space  $\Omega^1(X, \mathfrak{t})$ .  $\mathcal{G} := \text{Map}(X, \mathbb{T})$  is the group of gauge transformations and acts on  $\mathcal{A}_P$  in the standard way. That is, for  $g \in \text{Map}(X, \mathbb{T})$ , and  $A_P \in \mathcal{A}_P$ ,  $A_P \cdot g := A_P + g^*\vartheta$ , where  $\vartheta \in \Omega^1(\mathbb{T}, \mathfrak{t})$  denotes the Maurer-Cartan form on  $\mathbb{T}$ .  $\text{CS}_{X,P}(A_P)$  is the Chern-Simons functional of a  $\mathbb{T}$ -connection  $A_P$  on  $P \rightarrow X$  and we describe this presently. For any  $\mathbb{T}$ -connection  $A_P \in \mathcal{A}_P$ , we define an  $\text{SU}(N+1)$ -connection  $\hat{A}_P$  on an associated principal  $\text{SU}(N+1)$ -bundle,

$$(1) \quad \hat{P} = P \times_{\mathbb{T}} \text{SU}(N+1),$$

via,

$$\hat{A}_P|_{[p,h]} = \text{Ad}_{h^{-1}}(\iota_* \text{pr}_1^* A_P|_p) + \text{pr}_2^* \vartheta_h,$$

where  $\iota : \mathbb{T} \rightarrow \text{SU}(N+1)$  is inclusion as a maximal torus,  $\text{pr}_1 : P \times \text{SU}(N+1) \rightarrow P$  and  $\text{pr}_2 : P \times \text{SU}(N+1) \rightarrow \text{SU}(N+1)$  are the standard projections. Since for any three manifold  $X$ ,  $\hat{P}$  is trivializable, let  $\hat{s} : X \rightarrow \hat{P}$  be a global section. The definition we use for the Chern-Simons action,  $\text{CS}_{X,P}(A_P)$ , is as follows,

**Definition 1.** *The Chern-Simons action functional of a  $\mathbb{T}$ -connection  $A_P \in \mathcal{A}_P$  is defined by,*

$$(2) \quad \text{CS}_{X,P}(A_P) := \frac{1}{4\pi} \int_X \hat{s}^* \alpha(\hat{A}_P) \mod (2\pi\mathbb{Z}),$$

where  $\alpha(\hat{A}_P) \in \Omega^3(\hat{P}, \mathbb{R})$  is the Chern-Simons form of the induced  $\text{SU}(N+1)$ -connection  $\hat{A}_P \in \mathcal{A}_{\hat{P}}$ ,

$$(3) \quad \alpha(\hat{A}_P) := \text{Tr}(\hat{A}_P \wedge F_{\hat{A}_P}) - \frac{1}{6} \text{Tr}(\hat{A}_P \wedge [\hat{A}_P, \hat{A}_P]),$$

where  $\text{Tr} : \mathfrak{su}(N+1) \otimes \mathfrak{su}(N+1) \rightarrow \mathbb{R}$  denotes the standard Ad-invariant bilinear form in the  $(N+1)$ -dimensional representation.

We also define the following,

- $m_X := \frac{N}{2}(\dim H^1(X; \mathbb{R}) - 2 \dim H^0(X; \mathbb{R}))$ ,
- $A_P$  denotes a flat connection on a principal  $\text{U}(1)$ -bundle  $P$  over  $X$  with Chern-Simons invariant  $\text{CS}_{X,P}(A_P)$ ,
- $[g, n; (\alpha_1, \beta_1), \dots, (\alpha_M, \beta_M)]$ , for  $\gcd(\alpha_j, \beta_j) = 1$  are the Seifert invariants of a quasi-regular Sasakian manifold  $(X, \phi, \xi, \kappa, g)$ ,

- $\eta_0 = N \left( \frac{c_1(X)}{6} - 2 \sum_{j=1}^M s(\alpha_j, \beta_j) \right)$  is the *adiabatic eta invariant* of the quasi-regular Sasakian manifold  $(X, \phi, \xi, \kappa, g)$  introduced in [Nic00],
- $s(\alpha, \beta) := \frac{1}{4\alpha} \sum_{j=1}^{\alpha-1} \cot\left(\frac{\pi j}{\alpha}\right) \cot\left(\frac{\pi j \beta}{\alpha}\right) \in \mathbb{Q}$  is the classical Rademacher-Dedekind sum,
- The moduli space of flat abelian connections is given by,

$$\mathcal{M}_X \simeq \prod_{[P] \in \text{Tors } H^2(X, \Lambda)} \mathbb{T}^{2g},$$

and a particular component of  $\mathcal{M}_X$  corresponding to a particular bundle class  $[P] \in \text{Tors } H^2(X, \Lambda)$  is denoted as,

$$\mathcal{M}_P \simeq H^1(X, \mathfrak{t})/H^1(X, \Lambda) \simeq \mathbb{T}^{2g}.$$

- The eta-invariant for the odd signature operator,  $L^\circ$ , acting on  $\Omega^1(X, \mathfrak{t}) \oplus \Omega^3(X, \mathfrak{t})$ , is roughly defined by analytic continuation as a limit,

$$(4) \quad \eta(L^\circ) := \lim_{s \rightarrow 0} \sum_{\lambda \in \text{spec}^*(L^\circ)} \text{sgn}(\lambda) |\lambda|^{-s}.$$

The eta-invariant is an analytic invariant introduced by Atiyah, Patodi and Singer [APS75a] defined for an elliptic and self-adjoint operator. As in [APS75a, Prop. 4.20], we may remove some spectral symmetry and the eta invariant of  $L^\circ$  coincides with the eta invariant of the operator  $\star d$  restricted to  $\Omega^1(X, \mathfrak{t}) \cap \text{Im}(d\star)$ . Throughout, we will abuse notation slightly and write,

$$(5) \quad \eta(\star d) = \lim_{s \rightarrow 0} \sum_{\lambda \in \text{spec}^*(\star d)} \text{sgn}(\lambda) |\lambda|^{-s},$$

and replace  $L^\circ$  in the notation with  $\star d$ . We also recall that the expression for the sum,

$$(6) \quad \sum_{\lambda \in \text{spec}^*(\star d)} \text{sgn}(\lambda) |\lambda|^{-s},$$

is defined for large  $\text{Re}(s)$  and [APS75a] shows that it has a meromorphic continuation to  $\mathbb{C}$  that is analytic at 0. It therefore makes sense to take the limit as  $s \rightarrow 0$  in Eq. (5) and to define the eta-invariant  $\eta(\star d)$  as evaluation of this limit.

- $\eta_{\text{grav}}(g)$  denotes the eta-invariant for the operator  $\star d$  acting on  $\Omega^1(X, \mathbb{R})$ , so that,

$$(7) \quad \eta(\star d) = N \cdot \eta_{\text{grav}}(g),$$

where the eta invariant on the left hand side of (7) is defined on  $\Omega^1(X, \mathfrak{t})$  and  $N = \dim \mathbb{T}$ ,

•

$$(8) \quad \text{CS}_s(A^g) := \frac{1}{4\pi} \int_X s^* \text{Tr}(A^g \wedge dA^g + \frac{2}{3} A^g \wedge A^g \wedge A^g),$$

is the gravitational Chern-Simons term, where  $A^g$  the Levi-Civita connection and  $s$  a trivializing section of twice the tangent bundle of  $X$ . More explicitly, let  $H = \text{Spin}(6)$ ,  $Q = TX \oplus TX$  viewed as a principal  $\text{Spin}(6)$ -bundle over  $X$ ,  $g \in \Gamma(S^2(T^*X))$  a Riemannian metric on  $X$ ,  $\phi : Q \rightarrow \text{SO}(X)$  a principal bundle morphism, and  $A^{LC} \in \mathcal{A}_{\text{SO}(X)} := \{A \in (\Omega^1(\text{SO}(X)) \otimes \mathfrak{so}(3))^{\text{SO}(3)} \mid A(\xi^\sharp) = \xi, \forall \xi \in \mathfrak{so}(3)\}$  the Levi-Civita connection. Then  $A^g := \phi^* A^{LC} \in \mathcal{A}_Q := \{A \in (\Omega^1(Q) \otimes \mathfrak{h})^H \mid A(\xi^\sharp) = \xi, \forall \xi \in \mathfrak{h}\}$ .

An Atiyah-Patodi-Singer theorem, [APS75b, Prop. 4.19], says that the combination,

$$(9) \quad \eta_{\text{grav}}(\mathfrak{g}) + \frac{1}{3} \frac{\text{CS}(A^{\mathfrak{g}})}{2\pi},$$

is a topological invariant depending only on a two-framing of  $X$ . Recall that a two-framing is a choice of a homotopy equivalence class  $\Pi$  of trivializations of  $TX \oplus TX$ , twice the tangent bundle of  $X$ . Note that  $\Pi$  is represented by the trivializing section  $s : X \rightarrow Q$  above. The possible two-framings correspond to  $\mathbb{Z}$ . The identification with  $\mathbb{Z}$  is given by the signature defect defined by,

$$\delta(X, \Pi) = \text{sign}(M) - \frac{1}{6} p_1(2TM, \Pi),$$

where  $M$  is a 4-manifold with boundary  $X$  and  $p_1(2TM, \Pi)$  is the relative Pontrjagin number associated to the framing  $\Pi$  of the bundle  $TX \oplus TX$ . The canonical two-framing  $\Pi^c$  corresponds to  $\delta(X, \Pi^c) = 0$ .

We are now ready to make the following,

**Definition 2.** [McL12] *Let  $k \in \mathbb{Z}$  and  $X$  a closed, oriented three-manifold. The abelian Chern-Simons partition function,  $Z_{\mathbb{T}}(X, k)$ , is the quantity,*

$$(10) \quad Z_{\mathbb{T}}(X, k) = \frac{1}{|W|} \cdot \sum_{P \in \text{Tors } H^2(X, \Lambda)} Z_{\mathbb{T}}(X, P, k),$$

where  $|W| = (N+1)!$  is the order of the Weyl group for  $\text{SU}(N+1)$  and,

$$(11) \quad Z_{\mathbb{T}}(X, P, k) := k^{m_X} e^{ik \text{CS}_{X,P}(A_P)} e^{\pi i N \left( \frac{\eta_{\text{grav}}(\mathfrak{g})}{4} + \frac{1}{12} \frac{\text{CS}(A^{\mathfrak{g}})}{2\pi} \right)} \int_{\mathcal{M}_P} \sqrt{T_X},$$

where  $m_X = \frac{N}{2}(\dim H^1(X, \mathbb{R}) - 2 \dim H^0(X, \mathbb{R}))$ .

**Definition 3.** [McL12] *Let  $k \in \mathbb{Z}$ , and let  $(X, \phi, \xi, \kappa, \mathfrak{g})$  be a closed oriented quasi-regular Sasakian three-manifold with associated principal bundle structure,*

$$\begin{array}{ccc} \mathbb{U}(1) & \hookrightarrow & X \\ & \downarrow & \\ & \Sigma & \end{array}$$

where  $\Sigma$  is an orbifold such that  $X$  has associated Seifert invariants,

$$[g, n; (\alpha_1, \beta_1), \dots, (\alpha_M, \beta_M)],$$

for  $\gcd(\alpha_j, \beta_j) = 1$ . Define the symplectic abelian Chern-Simons partition function,

$$(12) \quad \bar{Z}_{\mathbb{T}}(X, k) = \frac{1}{|W|} \cdot \sum_{[P] \in \text{Tors } H^2(X, \Lambda)} \bar{Z}_{\mathbb{T}}(X, P, k),$$

where  $|W| = (N+1)!$  is the order of the Weyl group for  $\text{SU}(N+1)$  and,

$$(13) \quad \bar{Z}_{\mathbb{T}}(X, P, k) = k^{m_X} e^{ik \text{CS}_{X,P}(A_P)} e^{i\pi \left( \frac{N}{4} - \frac{1}{2} \eta_0 \right)} \int_{\mathcal{M}_P} K_X \cdot \omega_P,$$

where,  $m_X := \frac{N}{2}(\dim H^1(X; \mathbb{R}) - 2 \dim H^0(X; \mathbb{R}))$ ,  $K_X := \frac{1}{|c_1(X) \cdot \prod_i \alpha_i|^{N/2}}$ ,  $\omega_P := \frac{(\sum_{j=1}^g d\theta_j \wedge d\bar{\theta}_j)^{gN}}{(gN)!(2\pi)^{2gN}}$ .

The main motivation for this work is the conjectural equivalence of the rigorous topological invariants  $Z_{\mathbb{T}}(X, k)$  and  $\bar{Z}_{\mathbb{T}}(X, k)$ . Indeed, this work settles a conjecture that arose in the authors Ph.D. thesis [McL10]. We note that part of this conjectural equivalence is motivated by [JM09] which argues that  $\sqrt{T_X}$  is proportional to a specific scalar multiple of the natural unit symplectic volume form  $\omega \in \Omega^{2gN}(\mathcal{M}_X, \mathbb{R})$  by using the group structure on the moduli space  $\mathcal{M}_X$ ,

$$(14) \quad \sqrt{T_X} = C \cdot \left( \frac{1}{\sqrt{|\text{Tors } H^2(X, \Lambda)|}} \cdot \omega \right),$$

where  $0 \neq C \in \mathbb{R}$ . The conjecture made in [McL10] is that  $C = 1$  in (14) after an appropriate choice of base is made for the zeroth cohomology  $H^0(X, \mathfrak{t})$ , and when  $X$  admits a *regular* Sasakian structure. This conjecture is settled in this article in the more general case of a three manifold  $X$  that admits a *quasi-regular* Sasakian structure. One of the main results of this article is theorem 17. We are able to calculate the square-root of  $T_X$  explicitly as a specific scalar multiple of a natural symplectic volume form on the moduli space  $\mathcal{M}_X$  using a theorem of Rumin and Seshadri [RS11, Theorem 5.4]. We obtain the following,

**Theorem 4.** *Let  $(X, \phi, \xi, \kappa, g)$  be a closed quasi-regular Sasakian three-manifold. Then,*

$$(15) \quad (T_X^{\text{Ad}}(\rho))^{1/2} = \frac{(2\pi)^{-Ng}}{|c_1(X) \cdot \prod_i \alpha_i|^{N/2}} \left| \bigwedge \delta_{\text{dR}}^1(\nu^1) \right|^*,$$

where  $|\bigwedge \delta_{\text{dR}}^1(\nu^1)|^* : \bigwedge^{\max} H^1(X, \mathfrak{t}) \rightarrow \mathbb{R}^+$  is the volume form associated to the base given by  $\delta_{\text{dR}}^1(\nu^1)$ . Let,

$$(16) \quad (T_X^{\text{Ad}})^{1/2} : \mathcal{M} \rightarrow \left| \bigwedge^{\max} T^* \mathcal{M} \right|,$$

be the density form defined by,

$$(17) \quad (T_X^{\text{Ad}})^{1/2}(\rho) := (T_X^{\text{Ad}}(\rho))^{1/2},$$

with respect to the Zariski topology on  $T^* \mathcal{M}$ . Then we may write,

$$(18) \quad (T_X^{\text{Ad}})^{1/2} = \frac{1}{|c_1(X) \cdot \prod_i \alpha_i|^{N/2}} \cdot \omega,$$

where,

$$(19) \quad \omega := \frac{\Omega^{gN}}{(gN)!(2\pi)^{2gN}},$$

and,

$$(20) \quad \Omega := \sum_{1 \leq i \leq gN} d\theta_i \wedge d\bar{\theta}_i.$$

We also provide an explicit description of the moduli space of flat abelian connections  $\mathcal{M}_X$  on a closed, quasi-regular Sasakian three-manifold in proposition 10.

**Proposition 5.** *Given a closed, oriented Seifert three manifold  $X$  such that  $c_1(X) \neq 0$  (i.e. a Sasakian three manifold) then,*

$$\mathcal{M}_X \simeq \mathbb{T}^{2g} \times \text{Tors}(H^2(X, \Lambda)) \simeq \text{Hom}(\pi_1(X), \mathbb{T}),$$

where,  $|\text{Tors } H^2(X, \Lambda)| = \mathbf{d}^N = |c_1(X) \cdot \prod_{j=1}^M \alpha_j|^N$ .

We note that theorem 17 combined with proposition 10 leads to an explicit computation of the symplectic volume of the moduli space. Thus, we have the following,

**Corollary 6.** *Given a closed, oriented quasi-regular Sasakian three manifold  $(X, \phi, \xi, \kappa, g)$  with associated Seifert data,*

$$[g, n; (\alpha_1, \beta_1), \dots, (\alpha_M, \beta_M)],$$

*for  $\gcd(\alpha_j, \beta_j) = 1$ , the symplectic volume of the moduli space  $\mathcal{M}_X$  with respect to the symplectic volume form  $\sqrt{T_X} \in \Omega^{2gN}(\mathcal{M}_X, \mathbb{R})$  is given by,*

$$(21) \quad \int_{\mathcal{M}_X} \sqrt{T_X} = \sqrt{|\text{Tors } H^2(X, \Lambda)|} = \left| c_1(X) \cdot \prod_j \alpha_j \right|^{N/2},$$

*where  $c_1(X) = n + \sum_{j=1}^M \frac{\beta_j}{\alpha_j}$  is the first orbifold Chern number of  $(X, \phi, \xi, \kappa, g)$  and  $N = \dim \mathbb{T}$ .*

As a consequence of theorem 17 we obtain the following verification of the above conjecture,

**Corollary 7.** *Let  $k \in \mathbb{Z}$ , and let  $(X, \phi, \xi, \kappa, g)$  be a closed oriented quasi-regular Sasakian three-manifold with associated principal bundle structure,*

$$\begin{array}{ccc} \mathbb{U}(1) & \hookrightarrow & X \\ & \downarrow & \\ & \Sigma & \end{array}$$

*where  $\Sigma$  is an orbifold such that  $(X, \phi, \xi, \kappa, g)$  has associated Seifert invariants,*

$$[g, n; (\alpha_1, \beta_1), \dots, (\alpha_M, \beta_M)],$$

*for  $\gcd(\alpha_j, \beta_j) = 1$ . Then the magnitudes of  $Z_{\mathbb{T}}(X, k)$  and  $\bar{Z}_{\mathbb{T}}(X, k)$  agree identically,*

$$(22) \quad |Z_{\mathbb{T}}(X, k)| = |\bar{Z}_{\mathbb{T}}(X, k)|,$$

*and,*

$$(23) \quad |Z_{\mathbb{T}}(X, k)| = \frac{k^{m_X}}{|W|} \cdot \frac{\left| \sum_{[P] \in \text{Tors } H^2(X, \Lambda)} e^{ik \text{CS}_{X,P}(A_P)} \right|}{\sqrt{|\text{Tors } H^2(X, \Lambda)|}},$$

*where  $m_X := \frac{N}{2}(\dim H^1(X; \mathbb{R}) - 2 \dim H^0(X; \mathbb{R}))$ ,  $|W| = (N+1)!$  is the order of the Weyl group for  $\text{SU}(N+1)$  and  $\text{Tors } H^2(X, \Lambda)$  is the torsion part of  $H^2(X, \Lambda)$  with values in the integral lattice  $\Lambda$ .*

Lastly, we note that it would be of interest to compute the quantity,

$$(24) \quad \frac{\left| \sum_{[P] \in \text{Tors } H^2(X, \Lambda)} e^{ik \text{CS}_{X,P}(A_P)} \right|}{|W|},$$

explicitly in terms of Seifert data associated to a given quasi-regular Sasakian structure on  $X$ , and indeed it would be interesting to make an explicit computation of this quantity on a general closed three-manifold.

## 2. THE MODULI SPACE OF ABELIAN CONNECTIONS

In order to be more explicit in our computation of the abelian Chern-Simons partition function we study the moduli space of flat connections,  $\mathcal{M}_X$ , over a Sasakian manifold  $(X, \phi, \xi, \kappa, g)$  more closely in this section. One of the virtues of the abelian theory is that the moduli space of flat connections is relatively simple. Let  $\mathbb{T}$  denote a compact, connected abelian Lie group of dimension  $N$ ,  $\mathfrak{t}$  denote its Lie algebra and  $\Lambda \subset \mathfrak{t}$  the integral lattice. Let  $\text{Tors } H^2(X, \Lambda)$  denote the torsion subgroup of  $H^2(X, \Lambda)$ . In general, we have the following,

**Proposition 8.** [Man98] *Let  $X$  be a smooth three manifold and let  $\mathcal{M}_X$  be the moduli space of flat abelian  $\mathbb{T}$  connections on  $X$ . Then,*

- *There is a natural identification,*

$$\mathcal{M}_X = H^1(X, \mathbb{T}),$$

*and,*

- $|\pi_0(\mathcal{M}_X)| = |\text{Tors } H^2(X, \Lambda)|$  *and each connected component of  $\mathcal{M}_X$  is diffeomorphic to the torus  $H^1(X, \mathfrak{t})/H^1(X, \Lambda)$ .*

Our goal in this section is to obtain an expression for the number of components of  $\mathcal{M}_X$  in terms of Seifert data for a given Sasakian three manifold  $(X, \phi, \xi, \kappa, g)$ . The main result of this section is contained in Prop. (10) below. Note that a Sasakian three manifold  $(X, \phi, \xi, \kappa, g)$  admits a natural Seifert structure. For a general overview of the relevant geometric background see appendix A or [BG08]. For us, this will mean that  $X$  admits a locally free  $\mathbb{U}(1)$  action and can be written as a *non-trivial*  $\mathbb{U}(1)$ -bundle over an orbifold,

$$\begin{array}{ccc} \mathbb{U}(1) & \hookrightarrow & X \\ & \downarrow & \\ & \Sigma & \end{array}$$

where  $\Sigma = \{|\Sigma|, \mathcal{U}\}$  is an orbifold with underlying space  $|\Sigma|$  a surface of genus  $g$ .

**Remark 9.** *The key property of Sasakian manifolds for us in this section is that they are precisely the three manifolds with non-vanishing first orbifold Chern-class,  $c_1(X)$  [Sco83].*

Recall that the fundamental group of a Seifert manifold  $X$ ,  $\pi_1(X)$ , is generated by the following elements, [Orl72],

$$\begin{aligned} a_p, b_p, \quad p = 1, \dots, g, \\ c_j, \quad j = 1, \dots, M, \\ h, \end{aligned}$$

which satisfy the relations,

$$\begin{aligned} [a_p, h] = [b_p, h] = [c_j, h] &= 1, \\ c_j^{\alpha_j} h^{\beta_j} &= 1, \\ \prod_{p=1}^g [a_p, b_p] \prod_{j=1}^M c_j &= h^n. \end{aligned}$$

The generator  $h$  is associated to the generic  $\mathbb{U}(1)$  fiber over  $\Sigma$ , the generators  $a_p, b_p$  come from the  $2g$  non-contractible cycles on  $\Sigma$ , and the generators  $c_j$  come from the small one

cycles in  $\Sigma$  around each of the orbifold points  $p_j$ . The moduli space of flat connections over  $X$  can be realized as the space of homomorphisms from  $\pi_1(X)$  to  $\mathbb{T}$ . Consider,

$$\rho : \pi_1(X) \rightarrow \mathbb{T},$$

and let,  $C_j = \rho(c_j)$ ,  $B = \rho(h)$ . Since  $\mathbb{T}$  is abelian, the generating relations for  $\pi_1(X)$  translate into the following restrictions on  $\rho$ ,

$$C_j^{\alpha_j} B^{\beta_j} = 1, \quad j = 1, \dots, M,$$

$$\prod_{j=1}^M C_j \cdot B^{-n} = 1,$$

where,

$$(25) \quad C_j = \left[ e^{2\pi i \psi_j^1}, \dots, e^{2\pi i \psi_j^k}, \dots, e^{2\pi i \psi_j^N} \right] \in \mathbb{T}, \quad j = 1, \dots, M,$$

$$(26) \quad B = \left[ e^{2\pi i \psi_0^1}, \dots, e^{2\pi i \psi_0^k}, \dots, e^{2\pi i \psi_0^N} \right] \in \mathbb{T},$$

for some  $\psi_j^k \in [0, 1)$ ,  $j = 0, \dots, M$  and  $k = 1, \dots, N$ . Let  $K_{l,j}$  be the following  $(M+1) \times (M+1)$  matrix,

$$(27) \quad K = \begin{pmatrix} -n & 1 & 1 & \cdots & 1 \\ \beta_1 & \alpha_1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ \beta_{M-1} & \vdots & 0 & \alpha_{M-1} & 0 \\ \beta_M & 0 & \cdots & 0 & \alpha_M \end{pmatrix}$$

We condense the above restrictions and write,

$$(28) \quad \prod_{j=0}^M e^{2\pi i K_{l,j} \psi_j^k} = 1,$$

for each  $k = 1, \dots, N$ . For simplicity, let us assume that  $N = 1$  in the remainder. Let  $\mathbf{v} = (\psi_0, \psi_1, \dots, \psi_M) \in \mathbb{R}^{M+1}$ . Then  $\mathbf{v}$  solves the above equation, (28), if and only if,

$$(29) \quad K \cdot \mathbf{v} = \mathbf{w} \in \mathbb{Z}^{M+1}.$$

Let  $\mathbf{d} := |\det K| \in \mathbb{Z}$ . Observe that  $|\mathbf{d}| = |\det K| = \left| \left( \prod_{j=1}^M \alpha_j \right) \cdot \left( n + \sum_{j=1}^M \frac{\beta_j}{\alpha_j} \right) \right| = \left| \left( -\prod_{j=1}^M \alpha_j \right) \cdot c_1(X) \right| \neq 0$ , since  $\alpha_j \neq 0$ ,  $\forall 1 \leq j \leq M$ , and the orbifold first Chern number  $c_1(X) \neq 0$  by assumption (i.e. the Seifert structure on  $X$  is associated to a Sasakian structure). Thus,  $K$  is invertible and,

$$\mathbf{v} = K^{-1} \mathbf{w} = \frac{(\text{Cof } K)^T}{\mathbf{d}} \mathbf{w},$$

where the cofactor matrix  $\text{Cof } K \in M_{M+1}(\mathbb{Z})$ . We conclude that  $\mathbf{v}$  must necessarily satisfy,

$$\mathbf{v} \in \frac{1}{\mathbf{d}} \mathbb{Z}^{M+1},$$

if it is a solution of (28). Since solutions of (28) are defined only up to the integers, we may therefore take  $\mathbf{v} \in \frac{1}{\mathbf{d}} \mathbb{Z}_{\mathbf{d}}^{M+1}$ , where  $\mathbb{Z}_{\mathbf{d}} := \mathbb{Z}/(\mathbf{d} \cdot \mathbb{Z})$ , the set of integers modulo  $\mathbf{d}$ . We then



have a one to one correspondence between solutions  $\mathbf{v} \in \frac{1}{\mathbf{d}}\mathbb{Z}_{\mathbf{d}}^{M+1}$  and vectors  $\tilde{\mathbf{v}} = \mathbf{d}\mathbf{v} \in \mathbb{Z}_{\mathbf{d}}^{M+1}$  such that  $K\tilde{\mathbf{v}} = 0 \in \mathbb{Z}_{\mathbf{d}}^{M+1}$ . Viewing,

$$K : \mathbb{Z}_{\mathbf{d}}^{M+1} \rightarrow \mathbb{Z}_{\mathbf{d}}^{M+1},$$

as a  $\mathbb{Z}_{\mathbf{d}}$ -module homomorphism, then solutions of (28) are given exactly by  $\frac{1}{\mathbf{d}} \cdot \text{Ker}(K)$ . Observe that the last  $M$  columns of  $K$  form an independent set and therefore  $\text{rank}(K) \geq M$ . One can show that in fact  $\text{rank}(K) = M$ . Let,

$$\hat{\alpha}_i := \frac{1}{\alpha_i} \prod_{j=1}^M \alpha_j.$$

Then it is not hard to show that,

$$\mathbf{s} := \left[ -\prod_{j=1}^M \alpha_j, \hat{\alpha}_1 \beta_1, \dots, \hat{\alpha}_M \beta_M \right] \in \text{Ker}(K) \subset \mathbb{Z}_{\mathbf{d}}^{M+1},$$

and therefore the kernel is cyclicly generated by  $\mathbf{s} \neq 0$ ,

$$\text{Ker}(K) = \langle \mathbf{s} \rangle \simeq \mathbb{Z}_{\mathbf{d}}.$$

Since,

$$\mathcal{M}_X \simeq \mathbb{T}^{2g} \times \text{Tors}(H^2(X, \Lambda)) \simeq \text{Hom}(\pi_1(X), \mathbb{T}),$$

by Prop. (8) in general, we therefore obtain the following,

**Proposition 10.** *Given a closed, oriented Seifert three manifold  $X$  such that  $c_1(X) \neq 0$  (i.e. a Sasakian three manifold) then,*

$$\mathcal{M}_X \simeq \mathbb{T}^{2g} \times \text{Tors}(H^2(X, \Lambda)) \simeq \text{Hom}(\pi_1(X), \mathbb{T}),$$

where,  $|\text{Tors } H^2(X, \Lambda)| = \mathbf{d}^N = |c_1(X) \cdot \prod_{j=1}^M \alpha_j|^N$ .

### 3. REIDEMEISTER-RAY-SINGER TORSION AND SYMPLECTIC VOLUME

In this section we compute the square root of the Reidemeister-Ray-Singer torsion  $(T_X)^{1/2}$  as a symplectic volume form on the moduli space of flat abelian connections,

$$\mathcal{M}_X \simeq \prod_{[P] \in \text{Tors } H^2(X, \Lambda)} \mathbb{T}^{2g},$$

in the case that  $X$  admits a quasi-regular Sasakian structure,  $(\xi, \kappa, \phi, g)$ .

**Remark 11.** *We will abuse notation and write  $\mathcal{M}$  for a specific component of the moduli space corresponding to a bundle class  $[P] \in \text{Tors } H^2(X, \Lambda)$ . We will sometimes use the notation  $\mathcal{M}_P$  for a specific bundle class as well. The notation should be clear from the context. Also, we note that  $(T_X)^{1/2}$  is more naturally viewed as an element of a determinant line over  $\mathcal{M}$  and in order to compare  $(T_X)^{1/2}$  to a symplectic volume form we must choose a base for the zeroth cohomology of  $X$ . See remark 16 for further elaboration on this point. We indicate here that such a choice is assumed implicitly in what follows.*

Previously, [JM09] have shown that  $(T_X)^{1/2}$  is proportional to a specific symplectic volume form on  $\mathcal{M}$  in the case that  $X$  admits a *regular* Sasakian structure, or equivalently a principal  $\mathbb{U}(1)$ -bundle structure over a closed surface,  $\Sigma$ , of genus  $g$ . Let,

$$\Omega := \sum_{1 \leq i \leq g, 1 \leq j \leq N} d\theta_{i,j} \wedge d\bar{\theta}_{i,j},$$

be the standard symplectic form on  $\mathcal{M} \simeq \mathbb{T}^{2g}$ , where  $N = \dim \mathbb{T}$  is the dimension of the Lie group for gauge transformations. As noted above, we will assume  $\mathbb{T} = \mathbb{U}(1)$  in this section and eventually set  $N = 1$ . The natural symplectic volume form that we consider is defined as,

$$(30) \quad \omega := \frac{\Omega^{gN}}{(gN)!(2\pi)^{2gN}}.$$

Due to our considerations of the Chern-Simons path integral, it is more natural to consider  $\omega$  multiplied by  $1/|\text{Vol}(I)|$ , where,

$$\text{Vol}(I) := \left[ \int_X \kappa \wedge d\kappa \right]^{N/2} = [c_1(X)]^{N/2},$$

and  $I < \mathcal{G}$  is viewed as the isotropy subgroup of the gauge group of a given abelian connection  $A \in \mathcal{A}$ . [JM09] shows,

$$(31) \quad (T_X)^{1/2} = C \cdot \frac{1}{|\text{Vol}(I)|} \omega,$$

for some non-zero constant  $C \in \mathbb{R}^*$  when  $X$  admits a *regular* Sasakian structure. It was conjectured in [McL10] that indeed  $C = 1$  in Eq. (31) when a natural base is chosen for the zeroth cohomology. We study this conjecture in this section by computing the torsion explicitly in the more general case when  $X$  admits a *quasi-regular* Sasakian structure. We find that we may take  $C = 1$  in the case of a regular Sasakian structure after choosing a particular base for  $H^0(X, \mathbb{R})$  as in remark 16. In the general case of a quasi-regular Sasakian structure, we also find that  $C = 1$ , provided that  $|\text{Vol}(I)|$  is replaced by  $|c_1(X) \cdot \prod_i \alpha_i|^{N/2}$ , where

$$[g, n; (\alpha_1, \beta_1), \dots, (\alpha_M, \beta_M)], \quad \gcd(\alpha_j, \beta_j) = 1,$$

are the Seifert invariants corresponding to the quasi-regular Sasakian structure given. Of course, when  $X$  admits a *regular* Sasakian structure we have that  $|\text{Vol}(I)| = |c_1(X)|^{N/2}$ , since there are no exceptional fibers in this case. Our main result follows from [RS11, Theorem 5.4], where the torsion is computed on a closed quasi-regular Sasakian three-manifold twisted by a unitary representation  $\rho : \pi_1(X) \rightarrow \mathbb{U}(r)$ . A proof of the conjecture then follows by substituting some known special values of the Riemann-zeta function.

We present a definition of the analytic Ray-Singer torsion,  $T_X$ . We start by recalling the definition of the analytic Ray-Singer torsion. We will restrict ourselves to the  $\mathbb{U}(1)$  structure group case for simplicity. Before we make our definition we recall some standard notation. Let  $(M, g)$  be a closed oriented Riemannian manifold of dimension  $m$  and let  $\rho : \pi_1(M) \rightarrow \mathbb{U}(1)$  be a representation of the fundamental group of  $M$ . Recall that  $\rho$  corresponds to a flat principal  $\mathbb{U}(1)$  bundle  $P$  over  $M$  equipped with a flat connection  $A_\rho \in \mathcal{A}_P$ . Given a representation  $\chi : \mathbb{U}(1) \rightarrow \text{Aut } \mathbb{F}$ , where  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ , we obtain an associated line bundle  $\mathcal{E}_\chi := P \times_\chi \mathbb{F}$  in the usual way.

**Remark 12.** We will only need to assume  $\chi$  to be the standard representation on  $\mathbb{C}$  as in [RS73] and [RS11]. We see no reason to restrict to the standard representation in general, however. We also speculate on extending this theory to any field  $\mathbb{F}$  of arbitrary characteristic. We will not pursue this here. We will therefore state our definition of the torsion making this choice explicit. The choice of standard representation seems to be common in the literature and for us it natural to loosen this restriction. Also, it is important to note that the torsion  $T_X$  that shows up most naturally in abelian Chern-Simons theory is defined in terms of the adjoint representation  $\chi = \text{Ad} : \text{U}(1) \rightarrow \text{Aut}(\mathfrak{u}(1)) \simeq \mathbb{R}^\times$ . This will not concern us because we obtain our desired results for abelian Chern-Simons theory by restricting to the trivial representation  $\rho_0 : \pi_1(M) \rightarrow \text{U}(1)$  and working in the standard representation.

Let,

$$d_{A_\rho}^\chi : \Omega^q(M, \mathcal{E}_\chi) \rightarrow \Omega^{q+1}(M, \mathcal{E}_\chi),$$

denote the covariant derivative associated to  $A_\rho$  and let,

$$\Delta_q^\chi(\rho) := (d_{A_\rho}^\chi)^* d_{A_\rho}^\chi + d_{A_\rho}^\chi (d_{A_\rho}^\chi)^* : \Omega^q(M, \mathcal{E}_\chi) \rightarrow \Omega^q(M, \mathcal{E}_\chi),$$

denote the corresponding Laplacians. We make the following,

**Definition 13.** [RS73] Let  $M$  be a closed oriented Riemannian manifold of dimension  $m$  and let  $\rho : \pi_1(M) \rightarrow \text{U}(1)$  be a representation of the fundamental group of  $M$  and let  $\chi : \text{U}(1) \rightarrow \text{Aut } \mathbb{F}$  be a representation of  $\text{U}(1)$  (where  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ ). Let  $\Delta_q^\chi(\rho) : \Omega^q(M, \mathcal{E}_\chi) \rightarrow \Omega^q(M, \mathcal{E}_\chi)$  denote the Laplacians in the representation  $\chi$ . For each  $0 \leq q \leq m$ , let  $h^q$  be a preferred base for the cohomology group  $H^q(M, \rho)$ . Let  $\mathcal{H}^q(M, \rho)$  denote the harmonic  $q$ -forms and let  $\nu^q$  be an orthonormal base for  $\mathcal{H}^q(M, \rho)$ . Let  $\delta_{\text{dR}}^q : \mathcal{H}^q(M, \rho) \rightarrow H^q(M, \rho)$  denote the de Rham map, sending  $\nu^q$  into a base  $\delta_{\text{dR}}^q(\nu^q)$  for  $H^q(M, \rho)$ . Let  $[\delta_{\text{dR}}^q(\nu^q)/h^q]$  denote the determinant of the change of base map to the preferred base. The analytic Ray-Singer torsion is defined in this situation as,

$$(32) \quad T_M^\chi(\rho)\{(h^q)\} := \exp \left( \sum_{q=0}^m (-1)^q \left[ \frac{1}{2} q \zeta_q'(0) + \log |[\delta_{\text{dR}}^q(\nu^q)/h^q]| \right] \right),$$

where  $\zeta_q(s)$  is the zeta-function for  $\Delta_q^\chi(\rho)$  defined for  $\text{Re}(s) \gg 0$  by,

$$(33) \quad \zeta_q(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{tr}(e^{t\Delta_q} - \Pi_q) dt,$$

analytically continued to  $\mathbb{C}$  as usual, and  $\Pi_q : \Omega^q(M, \rho) \rightarrow \mathcal{H}^q(M, \rho)$  is orthogonal projection. It is also useful to define the scalar quantity,

$$(34) \quad T_M^{s,\chi}(\rho) := \exp \left( \frac{1}{2} \sum_{q=0}^m (-1)^q q \zeta_q'(0) \right),$$

which we also call the torsion.

Observe that the torsion  $T_M^\chi(\rho)$  is naturally viewed as an element of the determinant line,

$$|\det H^\bullet(\Omega(M, \mathbb{R}), d_{A_\rho}^\chi)^*| := \bigotimes_{j=0}^3 |\det H^j(\Omega(M, \mathbb{R}), d_{A_\rho}^\chi)^{(-1)^j}|.$$

Recall also the following description of the fundamental group of  $X$  in terms of its Seifert data, [Orl72].  $\pi_1(X)$  is generated by the following elements,

$$\begin{aligned} a_p, b_p, \quad p = 1, \dots, g, \\ c_j, \quad j = 1, \dots, M, \\ h, \end{aligned}$$

which satisfy the relations,

$$(35) \quad \begin{aligned} [a_p, h] = [b_p, h] = [c_j, h] &= 1, \\ c_j^{\alpha_j} h^{\beta_j} &= 1, \\ \prod_{p=1}^g [a_p, b_p] \prod_{j=1}^M c_j &= h^n. \end{aligned}$$

The generator  $h$  is associated to the generic  $S^1$  fiber over  $\Sigma$ , the generators  $a_p, b_p$  come from the  $2g$  non-contractible cycles on  $\Sigma$ , and the generators  $c_j$  come from the small one cycles in  $\Sigma$  around each of the orbifold points  $p_j$ . Before we quote the main result that we need we note that [RS11] uses the terminology ‘‘CR-Seifert’’ manifold, whereas we use the terminology ‘‘quasi-regular Sasakian’’ manifold. As shown in appendix A in proposition 51 the two notions are equivalent. We have the following,

**Theorem 14.** [RS11, Theorem 5.4] *Let  $(X, \phi, \xi, \kappa, g)$  be a closed quasi-regular Sasakian three-manifold. Split  $\mathcal{E}_\chi$  into irreducible  $\mathcal{E}_\chi^\theta$ , then the torsion function spectrally decomposes as,*

$$(36) \quad K(s) = \sum_{\mathcal{E}_\chi^\theta} K_\theta(s),$$

such that,

- On  $\mathcal{E}_\chi^\theta$  with  $\theta \in (0, 1)$ , i.e.  $\chi \circ \rho(h) = e^{2\pi i \theta} \neq 1$ , we have,

$$(37) \quad K_\theta(s) = \dim(\mathcal{E}_\chi^\theta) \chi(\Sigma^*) (\zeta(2s, \theta) + \zeta(2s, 1 - \theta)) +$$

$$(38) \quad + \sum_{i,j} \frac{1}{\alpha_i^{2s}} (\zeta(2s, \theta_{i,j}) + \zeta(2s, 1 - \theta_{i,j})).$$

- Let  $\mathcal{E}_\chi^{0,i} = \ker(1 - \chi \circ \rho(c_i))$ . Then we have,

$$\begin{aligned} K_0(s) &= K(X, \rho) (2\zeta(2s) + 1) + 2\zeta(2s) \sum_i \dim(\mathcal{E}_\chi^{0,i}) (\alpha_i^{-2s} - 1) + \\ &+ \sum_{\{(i,j): \theta_{i,j} \neq 0\}} \frac{1}{\alpha_i^{2s}} (\zeta(2s, \theta_{i,j}) + \zeta(2s, 1 - \theta_{i,j})). \end{aligned}$$

**Remark 15.** We note that the proof of this theorem relies heavily on the assumption that  $X$  admits a quasi-regular Sasakian structure. It is precisely the rigidity of the Sasakian structure that allows [RS11] to compare the spectra of the ordinary and horizontal Laplacians.

The case of interest for us is the trivial representation  $\rho_0 : \pi_1(X) \rightarrow \text{U}(1)$ . Since this is already scalar we have,

$$(39) \quad K(s) = K_0(s),$$

where, by Theorem (14), we have,

$$(40) \quad K_0(s) = K(X, \rho)(2\zeta(2s) + 1) + 2\zeta(2s) \sum_i (\alpha_i^{-2s} - 1).$$

Since  $T_X^{s,\chi}(\rho_0) = \exp(-K'_0(0)/2)$ , we compute  $K'_0(0)$ . Using the special values of the Riemann-zeta function,  $\zeta(0) = -1/2$  and  $\zeta'(0) = -\ln(2\pi)/2$  [Wil10], and  $K(X, \rho) = 2 \dim H^0(X, \mathfrak{t}) - \dim H^1(X, \mathfrak{t})$  [RS11, Eq. 42], we obtain,

$$(41) \quad -K'_0(0)/2 = (2 - 2g) \ln(2\pi) - \sum_i \ln(\alpha_i).$$

Thus,

$$(42) \quad T_X^{s,\chi}(\rho_0) = \frac{(2\pi)^{2-2g}}{\prod_i \alpha_i}.$$

It is easy to see that  $T_X^{s,\text{Ad}}(\rho) = T_X^{s,\chi}(\rho_0)$  when  $\rho_0 \equiv 1$  is the trivial representation,  $\chi$  is the standard representation, and  $\rho : \pi_1(X) \rightarrow \text{U}(1)$  is arbitrary. This follows because the spectra of the corresponding Laplacians are identical. That is, for the standard representation  $\chi$ , the Laplacian at the trivial representation  $\rho_0$  is given by,

$$\Delta_q^\chi(\rho_0) := d^*d + dd^* : \Omega^j(X, \mathbb{C}) \rightarrow \Omega^j(X, \mathbb{C}),$$

where  $d_{A_{\rho_0}}^\chi = d$  is just the ordinary de Rham derivative. Also, for the adjoint representation,

$$\Delta_q^{\text{Ad}}(\rho) := d^*d + dd^* : \Omega^j(X, \mathbb{R}) \rightarrow \Omega^j(X, \mathbb{R}),$$

since  $d_{A_\rho}^{\text{Ad}} = d$  for *any* representation  $\rho$ . Clearly, these operators have identical spectra.

By Poincaré duality  $H^3(\Omega(X, \mathbb{R}), d)'$  is canonically isomorphic to  $H^0(\Omega(X, \mathbb{R}), d)$ , and  $H^1(\Omega(X, \mathbb{R}), d)'$  is canonically isomorphic to  $H^2(\Omega(X, \mathbb{R}), d)$ . Thus,

$$T_X^{\text{Ad}}(\rho) \in |\det H^0(\Omega(X), d_{A_\rho})|^{\otimes 2} \bigotimes |\det H^1(\Omega(X), d_{A_\rho})'|^{\otimes 2},$$

and we naturally view,

$$(T_X^{\text{Ad}}(\rho))^{1/2} \in |\det H^0(\Omega(X), d_{A_\rho})| \bigotimes |\det H^1(\Omega(X), d_{A_\rho})'|.$$

**Remark 16.** Observe that  $\nu^0$  is an orthonormal base for  $\mathcal{H}^0(X, \mathbb{R}) = \mathbb{R}$ , and therefore is identified as a scalar  $\nu^0 \in \mathbb{R}$  such that,

$$\begin{aligned} 1 &= ||\nu^0||^2, \\ &= \int_X \nu^0 \wedge \star \nu^0, \\ &= |\nu^0|^2 \int_X \kappa \wedge d\kappa, \\ &= |\nu^0|^2 \cdot c_1(X), \end{aligned}$$

where  $c_1(X) = n + \sum \frac{\beta_j}{\alpha_j} > 0$  is the orbifold first Chern number of  $(X, \phi, \xi, \kappa, g)$  as a Seifert manifold with Seifert invariants,

$$[g, n; (\alpha_1, \beta_1), \dots, (\alpha_M, \beta_M)],$$

where  $\gcd(\alpha_j, \beta_j) = 1$ . Thus,  $|\nu^0| = 1/|c_1(X)|^{1/2}$ . In order to view the Reidemeister-Ray-Singer torsion as a volume form on  $\mathcal{M}$ , we must choose a base  $h^0$  for  $H^0(X, \mathbb{R})$  and

compute  $[\delta_{\text{dR}}^0(\nu^q)/h^0]$  as in definition (13). If we identify  $H^0(X, \mathbb{R}) \simeq \mathbb{R}$  via the de Rham map  $\delta_{\text{dR}}^0$ , then we choose  $h^0 = 1/2\pi$  under this identification so that  $|[\delta_{\text{dR}}^0(\nu^q)/h^0]| = 1/((2\pi) \cdot |c_1(X)|^{1/2})$ .

Overall we have the following,

**Theorem 17.** *Let  $(X, \phi, \xi, \kappa, g)$  be a closed quasi-regular Sasakian three-manifold. Then,*

$$(43) \quad (T_X^{\text{Ad}}(\rho))^{1/2} = \frac{(2\pi)^{-Ng}}{|c_1(X) \cdot \prod_i \alpha_i|^{N/2}} \left| \bigwedge \delta_{\text{dR}}^1(\nu^1) \right|^*,$$

where  $|\bigwedge \delta_{\text{dR}}^1(\nu^1)|^* : \bigwedge^{\max} H^1(X, \mathfrak{t}) \rightarrow \mathbb{R}^+$  is the volume form associated to the base given by  $\delta_{\text{dR}}^1(\nu^1)$ . Let,

$$(44) \quad (T_X^{\text{Ad}})^{1/2} : \mathcal{M} \rightarrow \left| \bigwedge^{\max} T^* \mathcal{M} \right|,$$

be the density form defined by,

$$(45) \quad (T_X^{\text{Ad}})^{1/2}(\rho) := (T_X^{\text{Ad}}(\rho))^{1/2},$$

with respect to the Zariski topology on  $T^* \mathcal{M}$ . Then we may write,

$$(46) \quad (T_X^{\text{Ad}})^{1/2} = \frac{1}{|c_1(X) \cdot \prod_i \alpha_i|^{N/2}} \cdot \omega,$$

where,

$$(47) \quad \omega := \frac{\Omega^{gN}}{(gN)!(2\pi)^{2gN}},$$

and,

$$(48) \quad \Omega := \sum_{1 \leq i \leq gN} d\theta_i \wedge d\bar{\theta}_i.$$

**Remark 18.** *Note that the generalization to the case of an arbitrary torus  $\mathbb{T}$  that occurs in proposition 17 is straightforward. We also point out that the extra factor of  $(2\pi)^{gN}$  that occurs in  $\omega$  in proposition (17) above is due to the corresponding factor of  $\sqrt{2\pi}$  in the norm of each orthonormal basis element for the first cohomology.*

## APPENDIX A. REVIEW OF SASAKIAN GEOMETRY

In this section we briefly review some definitions and results in contact geometry and also review the definition and some properties of Sasakian three-manifolds. We primarily follow [BG08]. Recall the following,

**Definition 19.** *A  $(2n+1)$ -dimensional manifold  $M$  is a contact manifold if there exists a one-form  $\kappa \in \Omega^1(M, \mathbb{R})$ , called a contact one-form, on  $M$ , such that,*

$$\kappa \wedge (d\kappa)^n \neq 0,$$

everywhere on  $M$ . A contact structure on  $M$  is an equivalence class of such one-forms, where  $\kappa' \sim \kappa \iff \exists 0 \neq f \in C^\infty(M)$  such that  $\kappa' = f\kappa$ . The subbundle  $\ker(\kappa) =: H \subset TM$  will be called the contact subbundle of  $M$ .

**Remark 20.** Note that the condition  $\kappa \wedge (d\kappa)^n \neq 0$  is equivalent to  $d\kappa$  being non-degenerate on  $H$ . There are several different perspectives and more general approaches to defining a contact structure on an odd dimensional manifold  $M$  that we will not pursue here [Bla76], [Gei08]. Let the line bundle  $\mathcal{L}_H$  be defined as the annihilator bundle of the contact subbundle  $\ker(\kappa) =: H \subset TM$ , i.e.  $\mathcal{L}_H := H^0$ . We only note that a more general definition involves allowing the line bundle  $\mathcal{L}_M \subset T^*M$  over  $M$  to be non-trivial. In Def. 19 we have assumed that  $\mathcal{L}_H$  is trivial and  $\kappa \in \Gamma(\mathcal{L}_H)$  represents a choice of trivializing section. In order to distinguish the two cases, we will refer to the case where  $\mathcal{L}_H$  is trivial as a strict (or co-orientable) contact structure, and the case where  $\mathcal{L}_H$  is non-trivial as a non-strict contact structure. The two-fold cover of a non-strict contact manifold is strict, and in particular, every simply connected contact manifold is strict.

**Example 21.** One of the most important examples of a contact manifold is  $\mathbb{R}^{2n+1}$  with contact form given by  $\kappa = dt - \sum_i y_i dx_i$ . The contact subbundle  $H$  is spanned by  $\{\frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial t}, \frac{\partial}{\partial y_i}\}$ . This clearly defines a contact structure on  $\mathbb{R}^{2n+1}$  according to Def. 19. For example  $d\kappa = dx_i \wedge dy_i$  is easily seen to be non-degenerate on  $H$ . This is called the standard contact structure on  $\mathbb{R}^{2n+1}$ . Note by the contact version of Darboux's theorem that every contact manifold is locally contactomorphic to  $\mathbb{R}^{2n+1}$  with the standard contact structure [Bla76]. Recall that a map  $\Psi : (M, \kappa) \rightarrow (M', \kappa')$  between contact manifolds is called a contactomorphism if it is a diffeomorphism that preserves the contact form,  $\Psi^* \kappa' = \kappa$ . If there exists a contactomorphism  $\Psi : (M, \kappa) \rightarrow (M', \kappa')$ , then  $(M, \kappa) \simeq (M', \kappa')$  are said to be contactomorphic.

**Example 22.** Let  $M = S^{2n+1}$ , the unit  $(2n+1)$ -sphere. Let  $\alpha := \sum_{i=0}^n (x_i dy_i - y_i dx_i) \in \Omega^1(\mathbb{R}^{2n+2}, \mathbb{R})$ , where  $\mathbb{R}^{2n+2}$  is given the standard Cartesian coordinates  $(x_0, \dots, x_n, y_0, \dots, y_n)$ . Define,

$$\kappa := \alpha|_{S^{2n+1}}.$$

It is straightforward to see that  $\kappa \wedge (d\kappa)^n \neq 0$  everywhere on  $S^{2n+1}$ . This defines the standard contact structure on  $S^{2n+1}$ .

There is a very interesting generalization of Ex. 22 above due to Gray [Gra59].

**Proposition 23.** [BG08, Prop. 6.1.17] Let  $M$  be an immersed hypersurface in  $\mathbb{R}^{2n+2}$  such that no tangent space of  $M$  contains the origin of  $\mathbb{R}^{2n+2}$ . Then  $M$  admits a contact structure.

It is interesting to note that the following provides an example of a contact manifold for which the results of this article do *not* apply.

**Example 24.** Let  $M = \mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$ . Let  $(x, y, z)$  denote the standard Cartesian coordinates in  $\mathbb{R}^3$  and let

$$\kappa := \sin(y)dx + \cos(y)dz.$$

Then  $\kappa \wedge d\kappa = -dx \wedge dy \wedge dz$ , and the contact subbundle is spanned by  $\{\frac{\partial}{\partial y}, \cos(y)\frac{\partial}{\partial x} - \sin(y)\frac{\partial}{\partial z}\}$ .

Recall that every oriented surface admits a *symplectic* structure. The analogue of this fairly trivial fact in the three-manifold case is the reason why studying contact structures on three-manifolds is so natural. We have the following,

**Theorem 25.** [Mar71] Every orientable three-manifold admits a contact structure.

We recall the following standard fact.

**Lemma 26.** *On a contact manifold  $(M, \kappa)$  there is a unique vector field  $\xi \in \Gamma(TM)$  called the Reeb vector field, satisfying the two conditions,*

$$\iota_\xi \kappa = 1, \quad \iota_\xi d\kappa = 0.$$

We note that the Reeb vector field  $\xi$  depends strongly on the choice of contact form  $\kappa$  and one can obtain very different Reeb vector fields for different choices of contact forms within a contact structure.

**Example 27.** *Consider  $(S^{2n+1}, \kappa)$  with the standard contact structure as in Ex. 22. Let  $H_i := x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i}$ . It is straightforward to see that the Reeb vector field of  $\kappa$  is given by,*

$$\xi = \sum_i H_i.$$

Let  $\mathbf{w} = (w_0, \dots, w_n) \in \mathbb{R}^{n+1}$  be some positive vector so that  $w_i > 0 \forall 0 \leq i \leq n$ . Let,

$$f_{\mathbf{w}}(x) := \frac{1}{\sum_{i=0}^n w_i (x_i^2 + y_i^2)}, \text{ for } x \in S^{2n+1}.$$

Define a deformed contact form,

$$\kappa_{\mathbf{w}} := f_{\mathbf{w}} \kappa.$$

It is easy to see that,

$$\xi_{\mathbf{w}} = \sum_i w_i H_i,$$

is the corresponding Reeb field for  $\kappa_{\mathbf{w}}$ . Clearly, the Reeb field changes drastically depending on the choice of vector  $\mathbf{w}$ . If the components of  $\mathbf{w}$  are rational numbers, for example, the orbits of the Reeb field turn out to be all circles. If we choose one of the components to be irrational, however, we may obtain Reeb orbits that do not close. Yet, since  $\kappa$  and  $\kappa_{\mathbf{w}} := f_{\mathbf{w}} \kappa$  differ by a non-zero function  $f_{\mathbf{w}}$ , these different choices amount to the same underlying contact structure.

The Reeb vector field is sometimes called the *characteristic vector field* and the one-dimensional foliation  $\mathcal{F}_\xi$  uniquely determined by  $\xi$  is called the *characteristic foliation* of  $(M, \kappa)$ .

**Definition 28.** *An almost contact structure on a differentiable manifold  $M$  is a triple  $(\xi, \kappa, \phi)$ , where  $\phi : TM \rightarrow TM$  is a tensor field of type  $(1, 1)$ ,  $\xi$  is a vector field, and  $\kappa \in \Omega^1(M, \mathbb{R})$  is a one-form which satisfy,*

$$\kappa(\xi) = 1, \quad \phi^2 = -\mathbb{I} + \xi \otimes \kappa,$$

where  $\mathbb{I}$  is the identity endomorphism of  $TM$ . A smooth manifold with such a structure is called an almost contact manifold. An almost contact structure is said to be normal if,

$$[\phi, \phi] + 2d\kappa \otimes \xi = 0,$$

where,

$$[\phi, \phi](Y_1, Y_2) := \phi^2[Y_1, Y_2] - [\phi Y_1, \phi Y_2] - \phi[\phi Y_1, Y_2] - \phi[Y_1, \phi Y_2],$$

is the Nijenhuis torsion of  $\phi$ .



**Definition 29.** Let  $M$  be an almost contact manifold. A Riemannian metric  $g$  on  $M$  is said to be compatible with the almost contact structure if for any vector fields  $Y_1, Y_2 \in \Gamma(TM)$ , we have

$$g(\phi Y_1, \phi Y_2) = g(Y_1, Y_2) - \kappa(Y_1)\kappa(Y_2).$$

An almost contact structure with a compatible metric is called an almost contact metric structure.

We have the following,

**Proposition 30.** [BG08] Every almost contact manifold admits a compatible metric.

We will need the following,

**Definition 31.** Let  $(M, \kappa)$  be a contact manifold with contact distribution  $H$ . Then an almost contact structure  $(\xi, \kappa', \phi)$  is said to be compatible with the contact structure if  $\kappa = \kappa'$ ,  $\xi$  is the Reeb vector field, and the endomorphism  $\phi$  satisfies,

$$d\kappa(\phi Y_1, \phi Y_2) = d\kappa(Y_1, Y_2), \text{ for all } Y_1, Y_2 \in \Gamma(TM),$$

and,

$$d\kappa(\phi Y_0, Y_0) > 0, \text{ for all } Y_0 \in \Gamma(H).$$

Denote by  $\mathcal{AC}(\kappa)$  the set of compatible almost contact structures on  $(M, \kappa)$ .

**Proposition 32.** [BG08, Prop. 6.4.3] Let  $(M, \kappa)$  be a contact manifold. The set of associated Riemannian metrics are in one-to-one correspondence with the set of compatible almost contact structures,  $\mathcal{AC}(\kappa)$ , on  $(M, \kappa)$ .

Finally, the following is the basic definition that we need for this article.

**Definition 33.** A contact manifold  $(M, \kappa)$  with a compatible almost contact metric structure  $(\xi, \kappa, \phi, g)$  such that,

$$g(Y_1, \phi Y_2) = d\kappa(Y_1, Y_2), \text{ for all } Y_1, Y_2 \in \Gamma(TM),$$

is called a contact metric structure, and  $(M, \xi, \kappa, \phi, g)$  is called a contact metric manifold.

**Definition 34.** A  $K$ -contact manifold is a manifold  $M$  with a contact metric structure  $(\phi, \xi, \kappa, g)$  such that the Reeb field  $\xi$  is Killing for the associated metric  $g$ ,  $\mathcal{L}_\xi g = 0$ .

**Definition 35.** The characteristic foliation  $\mathcal{F}_\xi$  of a contact manifold  $(M, \kappa)$  is said to be quasi-regular if there is a positive integer  $j$  such that each point has a foliated coordinate chart  $(U, x)$  such that each leaf of  $\mathcal{F}_\xi$  passes through  $U$  at most  $j$  times. If  $j = 1$  then the foliation is said to be regular.

Definitions (34) and (35) together define a quasi-regular  $K$ -contact manifold,  $(M, \phi, \xi, \kappa, g)$ . The following result provides several different perspectives on  $K$ -contact structures.

**Proposition 36.** [BG08, Prop. 6.4.8] On a contact metric manifold  $(M, \phi, \xi, \kappa, g)$ , the following conditions are equivalent:

- (i) The characteristic foliation  $\mathcal{F}_\xi$  is a Riemannian foliation.
- (ii)  $g$  is bundle-like.
- (iii) The Reeb flow is an isometry.
- (iv) The Reeb flow is a CR-transformation.
- (v) The contact metric structure  $(\phi, \xi, \kappa, g)$  is  $K$ -contact.

Recall the following,

**Definition 37.** A normal contact metric manifold  $(M, \xi, \kappa, \phi, g)$  is called a Sasakian manifold.

We will also need the following,

**Definition 38.** A Seifert manifold is a three-manifold  $X$  that admits a locally free  $\mathbb{U}(1)$ -action.

**Remark 39.** Note that our definition of a Seifert manifold is not the most general possible. We refer to [Orl72] for the general definition and also for the classification of these manifolds.

Thus, Seifert manifolds are simply  $\mathbb{U}(1)$ -bundles over an orbifold  $\Sigma$ ,

$$\begin{array}{ccc} \mathbb{U}(1) & \hookrightarrow & X \\ & \downarrow & \\ & \Sigma & \end{array}$$

It is well known that the topological isomorphism class of a Seifert manifold  $X$  is given by the Seifert invariants [Orl72],

$$[g, n; (\alpha_1, \beta_1), \dots, (\alpha_N, \beta_N)], \quad \gcd(\alpha_j, \beta_j) = 1,$$

where  $g$  is the genus of  $\Sigma$ . Geometrically, the  $\mathbb{U}(1)$  action on  $X$  is rotations of the fibres over  $\Sigma$  and the points in the  $\mathbb{U}(1)$  fiber over each orbifold point  $p_j$  on  $\Sigma$  are fixed by the cyclic subgroup  $\mathbb{Z}_{\alpha_j}$  of  $\mathbb{U}(1)$ . Recall also the following description of the fundamental group of  $X$ .  $\pi_1(X)$  is generated by the following elements [Orl72],

$$\begin{aligned} a_p, b_p, \quad p &= 1, \dots, g, \\ c_j, \quad j &= 1, \dots, M, \\ h, \end{aligned}$$

which satisfy the relations,

$$\begin{aligned} (49) \quad [a_p, h] &= [b_p, h] = [c_j, h] = 1, \\ c_j^{\alpha_j} h^{\beta_j} &= 1, \\ \prod_{p=1}^g [a_p, b_p] \prod_{j=1}^M c_j &= h^n, \end{aligned}$$

Geometrically, the generator  $h$  is associated to the generic  $\mathbb{U}(1)$  fiber over  $\Sigma$ , the generators  $a_p, b_p$  come from the  $2g$  non-contractible cycles on  $\Sigma$ , and the generators  $c_j$  come from the small one cycles in  $\Sigma$  around each of the orbifold points  $p_j$ .

Since every Sasakian three-manifold,  $(X, \phi, \xi, \kappa, g)$ , is K-contact [Bla76, Corollary 6.3] and every K-contact manifold admits a quasi-regular K-contact structure [BG08, Theorem 7.1.10], then every Sasakian three-manifold admits a quasi-regular K-contact structure,  $(X, \phi, \xi, \kappa, g)$ . We implicitly take a Sasakian three-manifold  $(X, \phi, \xi, \kappa, g)$  to be quasi-regular. Note that  $(X, \phi, \xi, \kappa, g)$  is a quasi-regular Sasakian three-manifold  $\iff$

- [BG08, Theorem 7.5.1, (i)]  $X$  is a Seifert manifold that is the total space of a principal  $\mathbb{U}(1)$  bundle over a Hodge orbifold surface,  $\Sigma$ .

- [BG08, Theorem 7.5.1, (iii)]  $X$  is a Seifert manifold that is the total space of a principal  $\mathbb{U}(1)$  bundle over a normal projective algebraic variety of real dimension two.

The following theorem is important for this article as it allows us to conclude a particularly nice form for the metric structures on Sasakian three-manifolds.

**Theorem 40.** [BG08, Theorem 6.3.6] *If  $(X, \phi, \xi, \kappa, g)$  is a quasi-regular Sasakian three-manifold, then the metric takes the form,*

$$g = \kappa \otimes \kappa + \pi^* h,$$

where  $h$  is a metric on the base,  $\Sigma$ , of  $X$ .

Sasakian geometry may also be viewed as an odd dimensional analogue of Kähler geometry. Let  $C(X) := \mathbb{R}^+ \times X$  be the cone on  $X$  with coordinate  $r$  on the  $\mathbb{R}^+$  factor and metric  $g_C := dr^2 + r^2 g$ . One may define a Sasakian structure  $(\kappa, \Phi, \xi, g)$  by requiring the associated structure on the metric cone  $(g_C, d(r^2 \kappa), J_C)$  to be Kähler. Note that trivial  $\mathbb{U}(1)$ -bundles over a surface  $\Sigma_g$ ,  $X = \mathbb{U}(1) \times \Sigma_g$ , admit no Sasakian structure [Ito97] and our results do not apply in this case.

**Example 41.** *All three dimensional lens spaces  $L(p, q)$  and the three sphere  $S^3$  admit quasi-regular Sasakian structures. Identify  $S^3 = \{z = (x, y) \in \mathbb{C}^2 : ||z||^2 = ||x||^2 + ||y||^2 = 1\}$ , and let,*

$$\begin{aligned} \kappa_0 &= \sum_{j=0}^1 (y_j dx_j - x_j dy_j) \Big|_{S^3}, \\ \xi_0 &= \sum_{j=0}^1 \left( y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right) \Big|_{S^3}, \\ \Phi_0 \Big|_{\ker \kappa_0} &= \sum_{j=0}^1 \left( \frac{\partial}{\partial y_j} \otimes dx_j - \frac{\partial}{\partial x_j} \otimes dy_j \right) \Big|_{\ker \kappa_0}, \quad \Phi_0(\xi_0) = 0, \\ g_0 &= \kappa_0 \otimes \kappa_0 + d\kappa_0 \circ (\mathbb{I} \otimes \Phi_0). \end{aligned}$$

Then  $(\Phi_0, \xi_0, \kappa_0, g_0)$  defines the standard Sasakian structure on  $S^3$ . This construction yields the Hopf fibration  $\mathbb{U}(1) \hookrightarrow S^3 \rightarrow S^2$ . One may obtain a Sasakian structure on the Lens space  $L(p, q)$  by taking the quotient of the standard Sasakian  $S^3$  by the usual Lens space  $\mathbb{Z}_p$  action.

Next, we note that [RS11] study “CR-Seifert” manifolds. Next we will show that a CR-Seifert structure naturally induces a quasi-regular Sasakian structure and the two structures are essentially equivalent for our purposes. First we review some CR geometry. We begin with the following,

**Definition 42.** [DT06, Def. 1.1 and 1.2] *An almost CR structure on a manifold  $X$ , with  $\dim X = m$ , is a subbundle  $T_{(1,0)} = T_{(1,0)}(X) \subset T^{\mathbb{C}}X$  of complex rank  $n$  of the complexified tangent bundle such that,*

$$T_{(1,0)}(X) \cap T_{(0,1)}(X) = 0,$$

where  $T_{(0,1)}(X) := \overline{T_{(1,0)}(X)}$  the complex conjugate. An almost CR structure is called a CR structure if,

$$[T_{(1,0)}(X), T_{(1,0)}(X)] \subset T_{(1,0)}(X),$$

so that  $T_{(1,0)}(X)$  is an integrable subbundle of  $T^{\mathbb{C}}X$ . The integers  $n$  and  $l = m - 2n$  are called the CR dimension and CR codimension of the almost CR structure and  $(n, l)$  denotes its type. The pair  $(X, T_{(1,0)})$  is called an (almost) CR manifold of type  $(n, l)$ .

We are mainly interested in almost CR structures of type  $(1, 1)$  in this article.

**Definition 43.** Let  $(X, T_{(1,0)})$  be an (almost) CR manifold of type  $(n, k)$ . The maximal complex, or Levi distribution is the real rank  $2n$  subbundle defined as,

$$L(X) = \Re(T_{(1,0)} \oplus T_{(1,0)}).$$

$L(X)$  carries the complex structure  $J_L : L(X) \rightarrow L(X)$  defined by,

$$J_L(Y + \bar{Y}) = i(Y + \bar{Y}),$$

for any  $Y \in T_{(1,0)}$ .

As noted in Remark (20) above, given a contact manifold  $(X, \kappa)$  with contact distribution  $H$ , a contact form is naturally viewed as a section of the annihilator bundle  $H^0$ . Generalizing this to CR manifolds of type  $(n, 1)$ , which are also called *CR manifolds of hypersurface type*, we let  $H^0$  denote the annihilator bundle of the Levi distribution  $H = L(X)$ . It is easy to see that  $H^0$  is a subbundle of  $T^*X$  that is isomorphic to  $TX/H$ . Assume  $X$  is orientable. Then since  $H$  is oriented by the complex structure  $J_L$ , it follows that  $H$  is orientable. Any orientable real line bundle over a connected manifold is trivial, so there exist globally defined nowhere vanishing sections  $\theta \in \Gamma(H)$ .

**Definition 44.** [DT06, Def. 1.6] Let  $(X, T_{(0,1)})$  be an oriented CR manifold of type  $(n, 1)$  with  $H = L(X)$ . Then any a choice of  $\theta \in \Gamma(H)$  is referred to as a pseudo-Hermitian structure on  $X$ . Given a pseudo-Hermitian structure  $\theta$  on  $X$  the Levi form  $L_\theta$  is defined by

$$L_\theta(Z, \bar{W}) = -id\theta(Z, \bar{W}),$$

for any  $Z, W \in T_{(1,0)}$ .

We now make the following,

**Definition 45.** [DT06, Def. 1.7] Let  $(X, T_{(0,1)})$  be an oriented CR manifold of type  $(n, 1)$  with  $H = L(X)$ . We say that  $(X, T_{(0,1)})$  is nondegenerate if the Levi form  $L_\theta$  is non-degenerate for some (and hence any) choice of pseudo-Hermitian structure  $\theta$  on  $X$ . If  $L_\theta$  is positive definite (i.e.  $L_\theta(Z, \bar{Z}) > 0, \forall 0 \neq Z \in T_{(0,1)}$ ) for some  $\theta$ , then  $(X, T_{(0,1)})$  is said to be strictly pseudoconvex. Of course, this does not apply to all choices of  $\theta$  since  $L_\theta$  positive definite implies that  $L_{-\theta}$  is negative definite.

Next we observe that one may obtain a natural contact metric structure  $(X, \xi, \kappa, \phi, g)$  from a CR structure  $(X, T_{(0,1)})$  of type  $(n, 1)$  with pseudo-Hermitian structure  $\kappa$ . First we need the following,

**Proposition 46.** [DT06, Prop. 1.2] Given  $(X, T_{(0,1)})$  a type  $(n, 1)$  CR manifold with pseudo-Hermitian structure  $\kappa$ , there exists a unique globally defined nowhere zero tangent vector field  $\xi$  on  $X$  such that,

$$\iota_\xi \kappa = 1, \quad \iota_\xi d\kappa = 0,$$

and  $\xi$  is transverse to the Levi distribution  $H = L(X)$ .

We also have,

**Proposition 47.** [DT06, Prop. 1.4] *Given  $(X, T_{(0,1)})$  a type  $(n, 1)$  CR manifold with pseudo-Hermitian structure  $\kappa$ , Levi distribution  $H = L(X)$  and  $\xi$  as in Prop. (46) above, then,*

$$TX \simeq H \oplus \mathbb{R}\xi.$$

By setting  $\phi(Y) = J_L Y$  for all  $Y \in H$ , and  $\phi\xi = 0$ , one can show that  $(X, \kappa, \xi, \phi)$  defines an almost contact manifold. If the Levi form  $L_\kappa$  is non-degenerate then  $(X, \kappa)$  is a contact manifold. Let,

$$g(Y_1, Y_2) = d\kappa(Y_1, J_L Y_2).$$

Then  $g(J_L Y_1, J_L Y_2) = g(Y_1, Y_2)$  since the Nijenhuis tensor of  $J_L$  vanishes when  $X$  is a CR manifold. We may now extend  $g$  to all of  $TX$  by using the splitting  $TX \simeq H \oplus \mathbb{R}\xi$  and defining  $g(Y, \xi) = 0$  and  $g(\xi, \xi) = 1$ . The resulting form  $g$  is called the *Webster metric* of  $(X, \kappa)$ . If the pseudo-Hermitian structure  $\kappa$  on  $X$  is strictly pseudoconvex, then  $g$  defines a Riemannian metric and  $(X, \xi, \kappa, \phi, g)$  defines a contact metric manifold. Consider the following,

**Example 48.** *Let  $(X, \kappa, \xi, \phi)$  be an almost contact manifold with distribution  $H = \ker \kappa$ . The restriction of  $\phi$  to  $H$  determines the decomposition,*

$$H^\mathbb{C} = H_{(1,0)} \oplus H_{(0,1)} \subset T^\mathbb{C}X,$$

where  $H_{(1,0)}$  and  $H_{(0,1)}$  are the  $+i$  and  $-i$  eigenbundles of  $J := \phi|_H$ , respectively. Taking  $T_{(1,0)}X = H_{(1,0)}$  determines an almost CR structure on  $X$ . This construction clearly also applies to the case where  $(X, \kappa)$  is a contact manifold and  $(X, \kappa, \xi, \phi)$  is choice of compatible almost complex structure. Whether or not this construction yields a CR structure (i.e. integrable distribution) has been determined by S. Tanno in [Tan89].

As observed in the above example 48 a given contact metric manifold  $(X, \xi, \kappa, \phi, g)$  does not always induce a (strongly pseudoconvex) CR structure. The case of dimension three is special, however, and we have the following,

**Proposition 49.** [Bla76, Corollary 6.4] *A three-dimensional contact metric manifold is a strongly pseudoconvex CR-manifold.*

We will see that that a “CR-Seifert” manifold naturally corresponds to a contact metric structure that is quasi-regular. We make the following,

**Definition 50.** *A CR-Seifert manifold is a three-dimensional compact manifold endowed with both a strictly pseudoconvex CR structure  $(H, \phi)$  and a Seifert structure that are compatible in the sense that the circle action  $\psi : \mathbb{U}(1) \rightarrow \text{Diff}(X)$  preserves the CR structure and is generated by a Reeb field  $\xi$ .*

Thus, given a CR-Seifert manifold as in definition 50 we obtain a natural contact metric structure associated by the above construction. It remains to show that this contact metric structure is indeed quasi-regular Sasakian. By proposition 36 we see that this contact metric structure must be  $K$ -contact since the circle action of the CR-Seifert structure acts by CR-transformations by definition and therefore this structure is Sasakian by [Bla76, Corollary 6.5]. By the results of Thomas [Tho76] we also know that the locally free circle action associated to the Seifert structure on  $X$  is equivalent to the Reeb foliation being quasi-regular. Conversely, given a quasi-regular Sasakian structure, we obtain a natural CR-Seifert structure and this correspondence is one-to-one. Thus, we have shown the following,

**Proposition 51.** *There is a natural one-to-one correspondence between the CR-Seifert structures and the quasi-regular Sasakian structures on a closed orientable three-manifold  $X$ .*

#### ACKNOWLEDGMENTS

This work represents an extension of part of my Ph.D. thesis and accordingly there are many people whom I would like to thank here. First and foremost, I would like to take this opportunity to thank my thesis advisor, Lisa Jeffrey, for her patience, wisdom, creativity and for sharing with me her encyclopedic knowledge of mathematics. This work would not have been possible without her. Among the many other people from whom I have benefited, I would particularly like to thank Jørgen Andersen, Dror Bar-Natan, Chris Beasley, John Bland, Vincent Bouchard, Ben Burrington, Stanley Deser, Dan Freed, Benjamin Himpe, Roman Jackiw, Yael Karshon, Eckhard Meinrenken, Raphaël Ponge, Frédéric Rochon, Michel Rumin, Paul Selick, Nicolai Reshetikhin, Vladimir Turaev, Jonathan Weitsman, and Edward Witten. This work was partially supported by a National Science and Engineering Research Council of Canada Graduate Scholarship Award, and the Danish National Research Foundation.

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